The Collatz 3n+1 Conjecture is Unprovable

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In this note, we consider the following function:

Definition 1: Let $T: \mathbb{N} \to \mathbb{N}$ be a function such that $T(n) = \frac{3n+1}{2}$ if n is odd and $T(n) = \frac{n}{2}$ if n is even.

The Collatz 3n+1 Conjecture states that for each $n \in \mathbb{N}$, there exists a $k \in \mathbb{N}$ such that $T^{(k)}(n)=1$, where $T^{(k)}(n)$ is the function T iteratively applied k times to n. As of September 4, 2003, the conjecture has been verified for all positive integers up to $224 \times 2^{50} \approx 2.52 \times 10^{17}$ (Roosendaal, 2003+). Furthermore, one can give a heuristic probabilistic argument (Crandall, 1978) that since every iterate of the function T decreases on average by a multiplicative factor of about $(\frac{3}{2})^{1/2}(\frac{1}{2})^{1/2}=(\frac{3}{4})^{1/2}$, all iterates will eventually converge into the infinite cycle $\{1,2,1,2,\ldots\}$, assuming that each $T^{(i)}$ sufficiently mixes up n as if each $T^{(i)}(n)$ mod 2 were drawn at random from the set $\{0,1\}$. However, the Collatz 3n+1 Conjecture has never been formally proven. In this paper, we show using Chaitin's notion of randomness (Chaitin, 1990) that the Collatz 3n+1 Conjecture can, in fact, never be formally proven, even though there is a lot of evidence for its truth. The underlying assumption in our argument is that a proof is composed of bits (zeroes and ones) just like any computer text-file. First, let us present a definition of "random".

Definition 2: We shall say that vector $\mathbf{x} \in \{0,1\}^k$ is random if \mathbf{x} cannot be specified in less than k bits in a computer text-file.

For instance, the vector $\mathbf{x} = [010101...010101] \in \{0,1\}^{10^6}$ is not random, since we can specify \mathbf{x} in less than one million bits in a computer text-file. (We just did.) However, the vector of outcomes of one million coin-tosses has a good chance of fitting our definition of "random", since much of the time the most compact way of specifying such a vector in a computer text-file is to list the results of each coin-toss, in which one million bits are necessary.

Theorem 1: For any vector $\mathbf{x} \in \{0,1\}^k$, there exists an $n \in \mathbb{N}$ such that $\mathbf{x} = (n, T(n), ..., T^{(k-1)}(n)) \mod 2$.

Proof: A proof of this can be found in "The 3x + 1 problem and its generalizations" (Lagarias, 1985).

Theorem 2: If $k, n \in \mathbb{N}$ and $T^{(k)}(n) = 1$, then in order to prove that $T^{(k)}(n) = 1$, it is necessary to specify the values of $(n, T(n), ..., T^{(k-1)}(n)) \mod 2$ in the proof.

Proof: The formula for $T^{(k)}(n)$ is determined by the values of $(n, T(n), ..., T^{(k-1)}(n))$ mod 2, and there is a one-to-one correspondence between all of the possible formulas for $T^{(k)}(n)$ and all of the possible values of $(n, T(n), ..., T^{(k-1)}(n))$ mod 2 (Lagarias, 1985); therefore, in order to prove that $T^{(k)}(n) = 1$, it is necessary

to specify the values of $(n, T(n), ..., T^{(k-1)}(n))$ mod 2 in the proof, since in order to prove that $T^{(k)}(n) = 1$, it is necessary to specify the formula for $T^{(k)}(n)$ in the proof.

Theorem 3: It is impossible to prove the Collatz 3n + 1 Conjecture.

Proof: Suppose that there exists a proof of the Collatz 3n+1 Conjecture, and let L be the number of bits in such a proof. Now, let $\mathbf{x} \in \{0,1\}^{L+1}$ be a random vector, as defined above. (It is not difficult to prove that at least half of all vectors in $\{0,1\}^{L+1}$ are random (Chaitin, 1990).) By Theorem 1, there exists an $n \in \mathbb{N}$ such that $\mathbf{x} = (n, T(n), ..., T^{(L)}(n))$ mod 2 and $T^{(L+1)}(n) = T^{(L)}(n)$ mod 2. Then $T^{(L)}(n) > 2$, so if $T^{(k)}(n) = 1$, then k > L. Hence, by Theorem 2 it is necessary to specify the values of $(n, T(n), ..., T^{(L)}(n))$ mod 2 in order to prove that there exists a $k \in \mathbb{N}$ such that $T^{(k)}(n) = 1$. But since $(n, T(n), ..., T^{(L)}(n))$ mod 2 is a random vector, at least L + 1 bits are necessary to specify $(n, T(n), ..., T^{(L)}(n))$ mod 2, contradicting our assumption that the proof contains only L bits; therefore, a formal proof of the Collatz 3n+1 Conjecture cannot exist.

Discussion: As we see, the core reason why the Collatz 3n+1 Conjecture is unprovable is because there is no limit to the number of bits that may be necessary to prove for a given $n \in \mathbb{N}$ that $T^{(k)}(n) = 1$ for some $k \in \mathbb{N}$.

Another famous conjecture that has a similar problem is, according to many, the most important unsolved problem in all of mathematics, the Riemann Hypothesis: Let the Riemann-Zeta function be a complex function defined by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ for $\Re(s) > 1$ and by analytic continuation elsewhere. It is well known that the only roots $\rho = \sigma + ti$ of ζ in which $\sigma \leq 0$ are $\rho = -2, -4, -6, ...$ and that there are no roots in which $\sigma \geq 1$. And also, there is a simple pole at s=1. The Riemann Hypothesis states that if $\rho = \sigma + ti$ is a root of ζ and $0 < \sigma < 1$, then $\sigma = 1/2$. It is well known that there are infinitely many roots of ζ that have $0 < \sigma < 1$. And just like the Collatz 3n+1 Conjecture, the Riemann Hypothesis has been verified by high-speed computers - for all |t| < T where $T \approx 2.0 \times 10^{20}$ (Odlyzko, 1989). But it is still unknown whether there exists a $|t| \geq T$ such that $\zeta(\sigma + ti) = 0$, where $\sigma \neq 1/2$. And just like the Collatz 3n+1 Conjecture, one can give a heuristic probabilistic argument (Good & Churchhouse, 1968) that the Riemann Hypothesis is true, as follows:

It is well known that the Riemann Hypothesis is equivalent to the statement that $M(n) = \sum_{k=1}^{n} \mu(k) = o(n^{1/2+\epsilon})$ for each $\epsilon > 0$, where μ is the Möbius Inversion function defined on $\mathbb N$ in which $\mu(k) = -1$ if k is the product of an odd number of distinct primes, $\mu(k) = 1$ if k is the product of an even number of distinct primes, and $\mu(k) = 0$ otherwise. Then if we are to assume that M(n) is distributed in the long run as a random walk, which is certainly plausible since the sequence has characteristics of a random walk, then by probability theory, $M(n) = o(n^{1/2+\epsilon})$ with probability one, for each $\epsilon > 0$.

From here on in this paper, whenever we say "number of roots", we shall mean "number of roots, counting multiplicities": By the argument principle, one can determine the number of roots of ζ in $\{s=\sigma+ti:\ 0<\sigma<1,\ 0< t< T\}$ by integrating $\frac{\zeta'(s)}{\zeta(s)}$ on the border of this domain minus, for small $\epsilon>0$, the horizontal strip $\{s=\sigma+ti:\ 0<\sigma<1,\ 0< t<\epsilon\}$ (in order to avoid the pole at s=1) and then dividing by $2\pi i$. However, determining whether these roots are all on the critical line $\sigma=1/2$ is a bit more tricky: The method by which this has been verified by high-speed computers is to consider a specific real function, Z(t), such that $|Z(t)|=|\zeta(1/2+ti)|$, so that the Riemann Hypothesis is equivalent to Z(t) having only real and pure imaginary roots. The computer determines a lower bound for the number of real roots of Z(t) in which 0< t< T by examining the sign changes in Z(t) (and if necessary the sign changes in its derivatives) and compares this number to a calculated upper bound for the number of roots of the ζ function in the domain, $\{s=\sigma+ti:\ 0<\sigma<1,\ 0< t< T\}$. If the numbers are equal, then the Riemann Hypothesis is verified for this domain (Pugh, 1998).

Now, let us observe the following, which will explain why there is a very good chance that the Riemann Hypothesis is unprovable, just like the Collatz 3n+1 Conjecture: Since there is no closed formula for the real roots of Z(t) (no formula that does not involve the function Z(t) like the formula for the roots $\rho = \sigma + ti = -2, -4, -6, \dots$ of ζ when $\sigma \leq 0$), the number of real roots of Z(t), where 0 < t < T, can only be

determined by examining the changes in sign of Z(t) and (if necessary) its derivatives; therefore, determining the number of real roots of Z(t) where 0 < t < T must involve computing the sign of Z(t) for various t. Let us assume that the time of the fastest algorithm that computes the sign of Z(t) approaches ∞ as $t \to \infty$, which is certainly a reasonable assumption, since the fastest known algorithm for doing such, the Riemann-Siegel formula (Odlyzko, 1994), runs in $O(\sqrt{t})$ time. Then proving that for each T > 0, the number of real roots of Z(t), where 0 < t < T, is equal to the number of roots of ζ in $\{s = \sigma + ti : 0 < \sigma < 1, 0 < t < T\}$ (which is equivalent to proving the Riemann Hypothesis) must take an infinite amount of time, since doing such requires computing the sign of Z(t) for arbitrarily large t and there is no limit to the number of computations that are necessary to determine such information. Hence, it appears that the Riemann Hypothesis is also unprovable, even though there is a lot of evidence for its truth.

Moral: If the Collatz 3n + 1 Conjecture is true, then only G-d can know this with absolute certainty. And this is also very likely to be the case with the Riemann Hypothesis.

Acknowledgements: I would like to thank G-d, my wife Aliza, my parents, my grandparents, and all my true friends for giving me the inspiration to write this paper. And I would like to give a special thanks to Peter Schorer and Donniell Fishkind for their help and encouragement and to Will Orrick and Anthony Quas for their helpful comments.

References:

Belaga, E. (1998), "Reflecting on the 3x + 1 Mystery: Outline of a scenario", U. Strasbourg preprint, 10 pages. http://www-irma.u-strasbg.fr/irma/publications/1998/98049.shtml

Chaitin, G.J., Algorithmic Information Theory, revised third printing, Cambridge University Press, 1990.

Crandall, R. E., "On the 3x + 1 problem", Math. Comp., 32 (1978) 1281–1292.

Good, I.J. and Churchhouse, R.F., "The Riemann Hypothesis and Pseudorandom Features of the Möbius Sequence", *Math. Comp.* 22 (1968), 857-861.

Lagarias, J.C., "The 3x + 1 problem and its generalizations", Amer. Math. Monthly 92 (1985) 3-23. [Reprinted in: Conference on Organic Mathematics, Canadian Math. Society Conference Proceedings vol 20, 1997, pp. 305-331], http://www.cecm.sfu.ca/organics/papers

Odlyzko, A.M., "Analytic computations in number theory", *Mathematics of Computation 1943-1993: A Half-Century of Computational Mathematics*, W. Gautschi (ed.), Amer. Math. Soc., Proc. Symp. Appl. Math. #48 (1994), pp. 451-463.

Odlyzko, A.M., "Supercomputers and the Riemann zeta function", Supercomputing 89: Supercomputing Structures & Computations, Proc. 4-th Intern. Conf. on Supercomputing, L.P. Kartashev and S.I. Kartashev (eds.), International Supercomputing Institute 1989, 348-352.

Pugh, G.R., "The Riemann-Siegel Formula and Large Scale Computations of the Riemann Zeta Function", Master's Thesis, University of British Columbia, 1998.

Roosendaal, E. (2003+), "On the 3x + 1 problem", http://www.ericr.nl/wondrous/index.html

Schorer, P., "The Structure of the 3x + 1 Function: An Introduction", http://www.occampress.com.